1

Metric Spaces: Basic Properties and Examples

1.1 INTRODUCTION

Metric space is an indispensable intermediate in course of evolution of the general topological spaces. It generalises the idea of distance between two points on the real line. Whenever we study the theory of functions of a real variable, the notion of distance between two real numbers intuitively comes. As for example, if we say $x \to a$, we mean that the absolute difference between a preassigned real number 'a' and the values the variable 'x' assumes, approaches zero—in mathematical notation, $|x - a| \to 0$. If one keeps in mind Cantor's geometric presentation of real numbers by points on a directed line, then the notation $|x - a| \to 0$ is viewed as equivalent to making distance between two points x and a on the real line tend to zero. Thus the idea of distance between two points on the real line plays a vital role in formulating the basic things like limit, continuity, differentiability, convergence in the real analysis.

Let's observe some notable properties of distance $|x - a|$ between two real numbers x and a. We agree to write $|x - a| \equiv d(x, a)$, a notation which we will carry to more generalised discussions coming up.

(P1) $d(x, a) \equiv |x - a| \geq 0$ (non-negativity) and
$d(x, a) = 0$ iff $|x - a| = 0$, *i.e.,* iff $x = a$ (positive-definiteness)

(P2) $d(x, a) \equiv |x - a| = |a - x| \equiv d(a, x)$ (symmetry)

(P3) $d(x, y) \equiv |x - y| = |x - a + a - y| \leq |x - a| + |a - y|$
$\equiv d(x, a) + d(a, y)$, (triangle inequality).

We now generalise the concept of 'distance' to an arbitrary non-empty set X, where distance function is defined in any way we like, the only constraint being the simultaneous satisfaction of the properties (P1), (P2), (P3) by it. Infact, we axiomatize the three properties, *viz,* non-negativity and positive-definiteness, symmetry and triangle inequality in the following definition:

Definition 1.1.1 Let X be a non-empty set and $d: X \times X \to R$ be a function that satisfies the conditions :

(d1) $d(x, y) \geq 0 \quad \forall x, y \in X$ and $d(x, y) = 0$ iff $x = y$

(d2) $d(x, y) = d(y, x)$ for $x, y \in X$

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$.

The function $d(\cdot, \cdot)$ satisfying (d1), (d2) and (d3) is called a *metric* and the structure (X, d) is called a *metric space*. Here we are not concerned with the specific objects (called points) of X and not even the specific rule of assignment $d(\cdot, .)$.

Note 1 If one defines $d: X \times X \to R^+ \cup \{0\}$, then the non-negativity property is redundant.

Note 2 Once we are convinced about the underlying metric d, we express (X, d) by mere X with the metric structure implied.

Note 3 The conditions which $d(\cdot, \cdot)$ satisfies just mimic the properties of the distance we are accustomed for real numbers, and hence these properties bear same names as their real-line counterparts.

Note 4. The non-negativity property of a metric is a consequence of its other properties as for any $x, y \in$ X, $0 = d(x, x) \le d(x, y) + d(y, x) = 2d(x, y)$

Note 5. In a metric space (X, d), $d(x_1, x_n) \le \sum_{i=1}^{n-1} d(x_i, x_{i+1})$ for any $x_1, x_2, \ldots\ldots x_n \in$ X.

It is an extension of triangle inequality and known as **polygonal inequality** (see exercise 5).

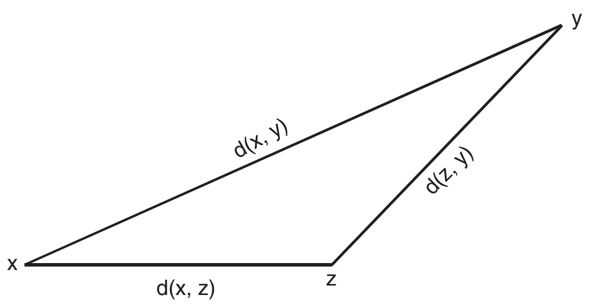

Fig 1.1.1 (a) Triangle Inequality

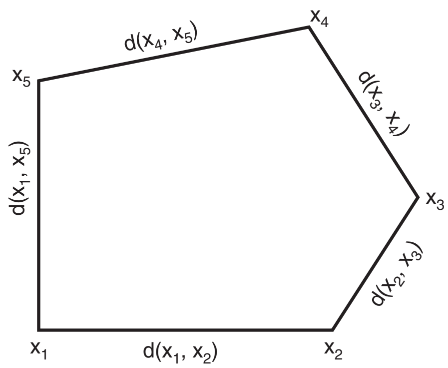

Fig. 1.1.1 (b) Polygonal Inequality (n = 5)

Note 6. At least for historical interest it is very curious that the motivation of introducing a metric in a function space evolved from classical brachistochrone problem of variational calculus. The brachistochrone problem, as we all know, deals with finding the shape of a smooth curve in a vertical plane along which heavy particle should slide under the action of gravity so that it consumes least time in traversing from a given point A to another given point B, A being sited higher than but not vertically above B. Thus we get a real valued function (time) defined on the family of smooth curves joining two given points A and B. If l be a specific curve of this family and $t(l)$ be the corresponding time of descent, brachistochrone problem aims at minimising $t(l)$ and hence find the curve of quickest descent.

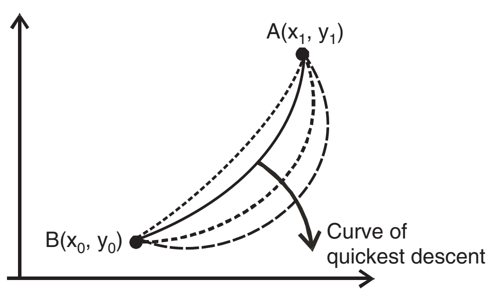

Fig. 1.1.2 Family of smooth curves (parameter l) joining A and B

Based on the difference in the time of descent one may be inclined to define a distance between two smooth curves joining two given points A and B and make foundation for defining a real-valued function, viz. metric, having space of curves as its domain.

Example 1.1.1 *Let $d: R \times R \to R$ be a function defined by $d(x, y) = \mid x - y \mid$, $x, y \in R$. To show that (R, d) is a metric space.*

Solution Since $\mid x - y \mid \geq 0$ and $\mid x - y \mid = 0$ iff $x = y$, (d1) follows. The other properties also follow as they are basically (P2) and (P3). Thus d is a metric on R and consequently (R, d) is a metric space. ■

As a prerequisite to the next example, we now state and prove Cauchy-Schwarz inequality:

If $\{p_1, p_2, ..., p_n\}$ and $\{q_1, q_2, ..., q_n\}$ be two sets of real numbers, then

$$\left(\sum_{i=1}^{n} p_i q_i\right)^2 \leq \left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{i=1}^{n} q_i^2\right)$$

Proof: (i) Let μ be any real number.

Define $$f(\mu) = \sum_{i=1}^{n} (p_i + \mu q_i)^2$$

$\therefore$ $f(\mu) \geq 0$ and is a quadratic in μ. If μ_1 and μ_2 be two distinct real roots of $f(\mu) = 0$, then we may write

$$f(\mu) = \sum_{i=1}^{n} (p_i + \mu q_i)^2 = \sum_{i=1}^{n} q_i^2 (\mu - \mu_1)(\mu - \mu_2)$$

Thus for any real t satisfying $\mu_1 < t < \mu_2$, $f(t)$ becomes negative contradicting the fact $f(\mu) \geq 0$ for all $\mu \in R$. Thus $f(\mu) = 0$, *i.e.,* the quadratic equation

$$\mu^2 \sum_{i=1}^{n} q_i^2 + 2\mu \sum_{i=1}^{n} p_i q_i + \sum_{i=1}^{n} p_i^2 = 0$$

cannot have two distinct real roots. This implies

$$\left(\sum_{i=1}^{n} p_i q_i\right)^2 - \left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{i=1}^{n} q_i^2\right) \leq 0 \qquad \text{(QED)}$$

(ii) $$0 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_i q_j - p_j q_i\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_i^2 q_j^2 + p_j^2 q_i^2 - 2p_i p_j q_i q_j\right)$$

$$= \left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{j=1}^{n} q_j^2\right) + \left(\sum_{j=1}^{n} p_j^2\right)\left(\sum_{i=1}^{n} q_i^2\right) - 2\sum_{i=1}^{n} p_i q_i \sum_{j=1}^{n} p_j q_j$$

$$= 2\left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{i=1}^{n} q_i^2\right) - 2\left(\sum_{i=1}^{n} p_i q_i\right)^2$$

(Since i and j are dummy indices of summation)

$$\therefore \qquad \left(\sum_{i=1}^{n} p_i q_i\right)^2 \le \left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{i=1}^{n} q_i^2\right) \qquad \text{(QED)}$$

Remark 1.1.1 Cauchy-Schwarz inequality may be deemed as an extension of the idea of dot product-norm relation $|\vec{a} \cdot \vec{b}|^2 \le |\vec{a}|^2 |\vec{b}|^2$ encountered in vector analysis.

Corollary. $$\left\{\sum_{i=1}^{n} (p_i + q_i)^2\right\}^{1/2} \le \left(\sum_{i=1}^{n} p_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} q_i^2\right)^{1/2}$$

Proof: $$\sum_{i=1}^{n} (p_i + q_i)^2 = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} q_i^2 + 2\sum_{i=1}^{n} p_i q_i$$

$$\le \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} q_i^2 + 2\left|\sum_{i=1}^{n} p_i q_i\right|$$

$$\le \left(\sum_{i=1}^{n} p_i^2\right) + \left(\sum_{i=1}^{n} q_i^2\right) + 2\left(\sum_{i=1}^{n} p_i^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i^2\right)^{1/2} \qquad \text{(by C.S. inequality)}$$

$$= \left[\left(\sum_{i=1}^{n} p_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} q_i^2\right)^{1/2}\right]^2$$

Taking the positive square root we get our desired result.

Example 1.1.2 *The n-dimensional Euclidean space* $\mathbf{R}^n$ *is a metric space with respect to the function* $d: R^n \times R^n \to R$, *defined by*

$$d(x, y) = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$$

where $x \equiv (x_1, x_2, ..., x_n)$ *and* $y \equiv (y_1, y_2, ..., y_n) \in R^n$, x_i, y_i*'s belonging to R.*

Solution: Obviously $d(x, y) \ge 0 \quad \forall\, x, y \in R,$

$$d(x, y) = 0 \quad \text{iff} \quad \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2} = 0 \quad \textit{i.e.,} \text{ iff } x_i = y_i \quad \forall\, i = 1, 2, ..., n.$$

Hence $x = y$ iff $d(x, y) = 0$.

Now let $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$ and $z \equiv (z_1, z_2, ..., z_n)$ be three arbitrary elements of R^n.

Since $x_i, y_i, z_i \in R \quad \forall\, i = 1, 2, ..., n$ and $p_i \equiv x_i - y_i$ and $q_i \equiv y_i - z_i \in R$, obviously $p_i + q_i = (x_i - z_i) \quad \forall\, i = 1, 2, ..., n$.

By the corollary we just proved,

$$\left\{\sum_{i=1}^{n}(p_i+q_i)^2\right\}^{1/2} \le \left(\sum_{i=1}^{n}p_i{}^2\right)^{1/2} + \left(\sum_{i=1}^{n}q_i{}^2\right)^{1/2}$$

i.e.,
$$\left\{\sum_{i=1}^{n}(x_i-z_i)^2\right\}^{1/2} \le \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i-z_i)^2\right)^{1/2}$$

i.e., $d(x, z) \le d(x, y) + d(y, z)$ (Triangle inequality)

Finally, $$d(x, y) = \left\{\sum_{i=1}^{n}(x_i-y_i)^2\right\}^{1/2} = \left(\sum_{i=1}^{n}(y_i-x_i)^2\right)^{1/2} = d(y, x), \text{ (symmetry).}$$

All these prove that $d(\cdot, \cdot)$ is a metric known popularly as **Euclidean Metric** or sometimes **Usual Metric.** ■

As a by product of this case we have the following example:

Example 1.1.3 *The Euclidean metric* $d: R^2 \times R^2 \to R$ *is defined by* $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, *where* $x \equiv (x_1, x_2) \in R^2$ *and* $y \equiv (y_1, y_2) \in R^2$. *The metric space* $\mathbf{R}^2 \equiv (R^2, d)$ *is called Euclidean plane. The proof is same in letter and spirit as the example 1.1.2.*

So on the same non-empty set X many metrics can be defined, as a result of which the same set X is endowed with different metric space structures. The following example is a nice illustration.

Example 1.1.4 *Let X be any non-empty set and d is a metric defined over X. Let m be any natural number so that we define $d_m(x, y) = md(x, y)$ for any $x, y \in X$. We are to show that $d_m(\cdot, \cdot)$ is also a metric. The new metric spaces $\{(X, d_m)/m = 1, 2, ...\}$ are thus obtained from (X, d).*

Solution: (*i*) $d_m(x, y) = md(x, y) \ge 0 \quad \forall\, x, y \in X$

Moreover $d_m(x, y) = 0$ iff $md(x, y) = 0$ *i.e.,* iff $d(x, y) = 0$ (since m is a natural number at our disposal), *i.e.,* iff $x = y$.

(*ii*) $d_m(x, y) = d_m(y, x)$ since $d(x, y) = d(y, x)$

(*iii*) $$\begin{aligned} d_m(x, y) &\equiv md(x, y) \\ &\le m(d(x, z) + d(z, y)) \\ &= md(x, z) + md(z, y) \\ &\equiv d_m(x, z) + d_m(z, y), \text{ for any } x, y, z \in X. \end{aligned}$$

Hence properties (d1) – (d3) are satisfied by $d_m(\cdot, \cdot)$. This metric is called ***dilation metric***.

Remark 1.1.2 The choice of m being a natural number has no specific advantage. However for $m > 1$, a 'dilation' and for $0 < m < 1$, a 'contraction' of distance occurs.

Example 1.1.5 *The set R^n is also a metric space with respect to another metric defined by*

$$d^*(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \text{ where } x \equiv (x_1, x_2, \dots x_n),\ y \equiv (y_1, y_2, \dots, y_n),\ x_i, y_i \in R,\ i = 1(1)\, n.$$

Solution: The conditions $(d1)$ and $(d2)$ are straight forward as in example 1.1.3. For $(d3)$, let $x, y, z \in R^n$ where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$ and $z \equiv (z_1, z_2, ..., z_n)$; $x_i, y_i, z_i \in R$, $i = 1, 2, ..., n$.

$$\text{Further} \quad d^*(x, z) = \sum_{i=1}^{n} | x_i - z_i | = \sum_{i=1}^{n} | x_i - y_i + y_i - z_i |$$

$$\leq \sum_{i=1}^{n} \{| x_i - y_i | + | y_i - z_i |\} = \sum_{i=1}^{n} | x_i - y_i | + \sum_{i=1}^{n} | y_i - z_i |$$

$$= d^*(x, y) + d^*(y, z)$$

Thus (R^n, d^*) is a metric space.

The metric d^* is called the **rectangular metric** on R^n. ■

Earlier we have shown in example 1.1.3 that the function d_1 defined by $d_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ with $x \equiv (x_1, x_2)$, $y \equiv (y_1, y_2)$ is a distance function. Again it readily follows from example 1.1.5 that $d_2(x, y) = | x_1 - y_1 | + | x_2 - y_2 |$ is also a distance function on R^2. We are interested in scanning the distance functions d_1 and d_2 from the geometrical point of view.

The metric d_2 (.,.) is known as **Taxicab metric** in R^2 as it measures the distance a taxi would travel from a point $A(x_1,x_2)$ to some other point $B(y_1,y_2)$ if there were no one way streets. Taxicab metric or its generalisation, viz, rectangular metric geometrically presents the sum of projections of the standard Euclidean distance [c.f. example 1.1.2 and 1.1.3] on the co-ordinate axes.

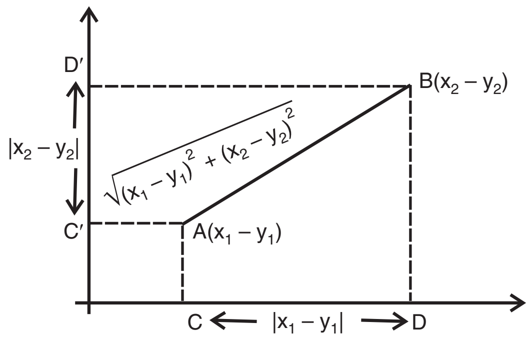

Fig. 1.1.3 Representation of Euclidean metric, Taxicab metric and Chebyshev metric in the backrop of R^2.

The rectangular metric is used in communication theory under the name "Hamming distance" that measures the discrepancy between two digital messages. It was introduced by R. Hamming (1950). (By a digital message of length n we mean a n-component column vector of 0's and 1's). The Hamming distance between two digital messages of same length is defined to be the number of co-ordinates in which they differ. So if $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ be any two digital messages of length n, *i.e.* x_i's and y_i's are only 0's and 1's, their Hamming distance $d_H(x, y)$ is given by $\sum_{i=1}^{n} | x_i - y_i |$. If there is a single discrepancy between the sent and received digital messages, their Hamming distance is unity. Thus Hamming distance is a metric on the set of all digital messages of a preassigned length.

Again consider the semi circular path with AB as diameter. Then obviously it will pass through C.

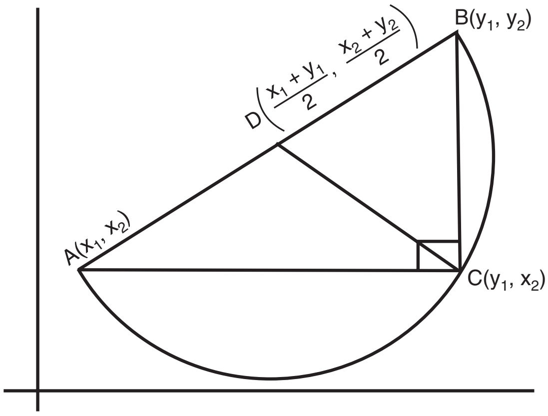

Fig. 1.1.4 Semicircular path joining two points

Clearly $D\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$, the mid point of AB, is the centre of the semicircular path and its length is

$$\pi \,.\, DC = \pi\sqrt{\left(\frac{x_1+y_1}{2}-y_1\right)^2+\left(\frac{x_2+y_2}{2}-x_2\right)^2}$$

$$= \frac{\pi}{2}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \frac{\pi}{2}\,.\,d_1(x, y)$$

Since $d_1(x, y)$ is a distance function on R^2 and $\dfrac{\pi}{2} > 1$, by example 1.1.4, it follows that $d_3(x, y) = \dfrac{\pi}{2}\, d_1(x, y)$ is also a distance function.

Thus we observe that upon the same non-empty set R^2, one can define more than one distance function or metric; it might be the straight linear distance or a broken-line distance or even a semicircular arcual distance. So whenever we talk of a metric space over R^2, we must keep in mind what specific kind of distance we are thinking.

Example 1.1.6 *The set R^n is a metric space with respect to the metric defined by $d(x, y) = Max.\{\mid x_i - y_i \mid ; i = 1, 2, ..., n\}$ where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$, $x_i, y_i \in R$, $i = 1, 2, ..., n$.*

Solution: Since $\mid x_i - y_i \mid \geq 0 \quad \forall\, i = 1, 2, ..., n$, Max. $\{\mid x_i - y_i \mid ; i = 1, 2, ... n\} \geq 0$. Again if $x = y$, then $x_i = y_i \;\forall\, i = 1, 2, ..., n$.

So $\mid x_i - y_i \mid = 0 \;\forall\, i = 1, 2, ..., n$ and hence Max. $\{\mid x_i - y_i \mid ; i = 1, 2, ..., n\} = 0$; *i.e.,* $d(x, y) = 0$. On the other hand if $d(x, y) = 0$ then Max. $\{\mid x_i - y_i \mid ; i = 1, 2, ..., n\} = 0$

$\Rightarrow \qquad | x_i - y_i | = 0 \quad \forall\, i = 1, 2, ..., n$, since each $| x_i - y_i | \geq 0$

$\therefore \qquad x = y.$

Thus $d(x, y) = 0$ if and only if $x = y$.

Next let $x, y \in R^n$ where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$ with $x_i, y_i \in R$, $i = 1, 2, ..., n$.

Then $d(x, y) = \text{Max.}\ \{| x_i - y_i |; i = 1, 2, ..., n\} = \text{Max.}\ \{| y_i - x_i |;\ i = 1, 2, ..., n\} = d(y, x)$.

Finally for $x, y, z \in R^n$, where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$,

$z \equiv (z_1, z_2, ..., z_n)$, and $x_i, y_i, z_i \in R$, $i = 1, 2, ..., n$.

$$\begin{aligned} d(x, z) &= \text{Max.}\ \{| x_i - z_i |, i = 1, 2, ..., n\} \\ &= \text{Max.}\ \{| x_i - y_i + y_i - z_i |;\ i = 1, 2, ..., n\} \\ &= \text{Max.}\ \{| x_i - y_i | + | y_i - z_i |; i = 1, 2, ..., n\} \\ &= \text{Max.}\ \{| x_i - y_i |; i = 1, 2, ..., n\} + \text{Max.}\ \{| y_i - y_i |; i = 1, 2, ..., n\} \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus d is a metric on R^n and hence (R^n, d) is also metric space. ■

Remark 1.1.3 In example (1.1.6), the Chebyshev metric presents the maximum of the projections of the standard Euclidean distance on the co-ordinate axes (see fig 1.1.3). The metric d above is known as **Chebyshev metric.**

Example 1.1.7 *The set R of real numbers is a metric space with respect to the metric defined by*

$$d(x, y) = Min.\ \{1,\ | x - y |\},\ x, y \in R.$$

Solution: Since $| x - y | \geq 0 \quad \forall\, x, y \in R$, Min. $\{1, | x - y |\} \geq 0$

Also if $x = y$, then Min. $\{1, | x - y |\}$ = Min. $\{1, 0\} = 0$

Again if Min. $\{1, | x - y |\} = 0$, then $| x - y | = 0$ which implies $x = y$. Thus $d(x, y) = 0$, if and only if $x = y$.

Next $d(x, y)$ = Min. $\{1, | x - y |\}$ = Min. $\{1, | y - x |\} = d(y, x)$.

Finally let $x, y, z \in R$. So $d(x, z)$ = Min. $\{1, |x - z|\}$

If Min. $\{1, | x - z |\} = 1$ then as $| x - z | \leq | x - y | + | y - z |$,

Min $\{1, | x - z |\}$ = Min $\{1, (| x - y | + | y - z |)\} \leq$ Min. $\{1, | x - y |\}$ + Min. $\{1, | y - z|\}$

Again if Min $\{1, | x - z |\} = | x - z |$, then also

Min. $\{1, | x - z |\} \leq$ Min $\{1, | x - y |\}$ + Min $\{1, | y - z |\}$

$\therefore$ Under all circumstances, $d(x, z) \leq d(x, y) + d(y, z)$

Thus d is a metric on R and hence (R, d) is a metric space. ■

Remark 1.1.4 If (X, d) be any metric space, then it is easy to prove that $d_1(. , .)$ defined by $d_1(x, y)$ = Min $\{1, d(x, y)\}\ \forall\, x, y \in X$ is also a metric on X.

This metric is known as **standard bounded metric** on X. In fact, corresponding to any metric $d(. , .)$ there always exists a metric $d_1(. , .)$ defined above. In the next chapter we shall see that in this metric space (X, d_1) every subset is bounded. One can generalise the definition of a bounded metric corresponding to $d(. , .)$ as $d_K(. , .)$ where

$$d_K(x, y) = \text{Min}\ \{K, d(x, y)\}\ \forall\, x, y \in X \text{ with } K > 0$$

Example 1.1.8 *Let $C[a, b]$ be the set of all real-valued continuous functions over $[a, b]$. Then $C[a, b]$ is a metric space with respect to the metric defined by*

$$d(f, g) = \sup_{u \in [a,b]} | f(u) - g(u) |,\ f, g \in C[a, b]$$

Solution: According to the definition, $d(f, g) \geq 0$. Further if $f = g$ then $f(u) = g(u)\ \forall\, u \in [a, b]$.

Therefore $| f(u) - g(u) | = 0\ \forall\, u \in [a, b]$ and hence $\sup_{u \in [a,b]} | f(u) - g(u) | = 0$. On the other hand if $d(f, g) = 0$ then $\sup_{u \in [a,b]} | f(u) - g(u) | = 0$. This means $| f(u) - g(u) | = 0$ for all $u \in [a, b]$ so that $f = g$. Thus $d(f, g) = 0$ if and only if $f = g$.

Next let $f, g \in C[a, b]$.

Then, $$d(f, g) = \sup_{u \in [a,b]} | f(u) - g(u) | = \sup_{u \in [a,b]} | g(u) - f(u) | = d(g, f).$$

Finally let $f, g, h \in C[a, b]$. Then $\forall\, u \in [a, b]$,

$$\begin{aligned} | f(u) - h(u)| &\le | f(u) - g(u)| + | g(u) - h(u)| \\ &\le \sup_{u \in [a,b]} | f(u) - g(u)| + \sup_{u \in [a,b]} | g(u) - h(u)| \\ &= d(f, g) + d(g, h). \end{aligned}$$

Taking supremum over $[a, b]$, we get $\sup_{u \in [a,b]} | f(u - h(u)| \le d(f, g) + f(g, h)$

$\therefore$ $$d(f, h) \le d(f, g) + d(g, h).$$

Thus d satisfies all the conditions ($d1$) to ($d3$) making $(C[a, b], d)$ a metric space. ■

This metric d is called the **sup metric** on $C[a, b]$.

In example (1.1.8), the so called supmetric or uniform metric geometrically presents maximum pointwise separation between two continuous functions f and g defined over $[a, b]$.

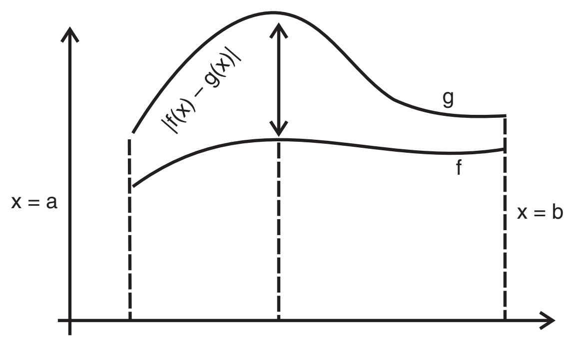

Fig. 1.1.5 Representation of Supmetric of example (1.1.8)

Example 1.1.9 *$C[a, b]$ is also a metric space with respect to the metric defined by*

$$d^*(f, g) = \int_a^b | f(u) - g(u) |\, du \text{ for } f, g \in C[a, b].$$

Solution: Since $| f(u) - g(u) |$ is also continuous for $f, g \in C[a, b]$, it is integrable over $[a, b]$. So the definition is meaningful. Since $| f(u) - g(u) |$ is non-negative, $d^*(f, g) \ge 0$ for all $f, g \in C[a, b]$. Further if $f = g$ then $| f(u) - g(u) | = 0$ $\forall\, u \in [a, b]$ and consequently $d^*(f, g) = 0$. Again if $\int_a^b | f(u) - g(u) |\, du = 0$ then since $| f(u) - g(u) |$ is non-negative and continuous on $[a, b]$, $| f(u) - g(u) | = 0$ $\forall\, u \in [a, b]$ which implies $f = g$.

Next let $f, g \in C[a\ \ b]$.

$\therefore \quad d^*(f, g) = \int_a^b | f(u) - g(u) | \, du = \int_a^b | g(u) - f(u) | \, du = d^*(g, f).$

Finally if $f, g, h \in C[a, b]$, then for all $u \in [a, b]$

$$| f(u) - h(u) | \leq | f(u) - g(u) | + | g(u) - h(u) |$$

$$\therefore \quad \int_a^b | f(u) - h(u) | \, du \leq \int_a^b \{f(u) - g(u) | + | g(u) - h(u) |\} \, du$$

$$= \int_a^b | f(u) - g(u) | \, du + \int_a^b | g(u) - h(u) | \, du$$

$$\therefore \quad d^*(f, h) \leq d^*(f, g) + d^*(g, h).$$

Thus d^* is a metric and $(C[a, b], d^*)$ is a metric space. This metric d^* is called the **Integral metric** on $C[a, b]$. ■

In example (1.1.9), the intergral metric represents the absolute area squeezed between two continuous functions f and g over the interval $[a, b]$.

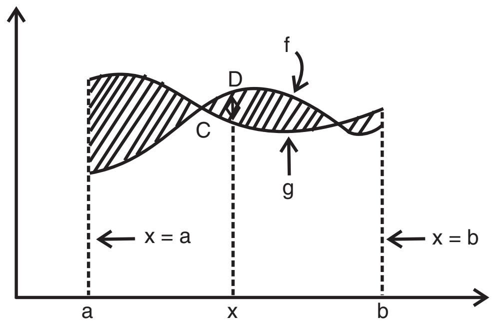

Fig. 1.1.6 Representation of Integral metric given in example (1.1.9)

Remark 1.1.4 If we define the integral $d^*(f, g)$ on $R[a, b]$, the set of all R-integrable functions over $[a, b]$, then $d^*(f, g)$ will not be a metric on $R[a, b]$. In fact $d^*(f, g) = 0$ does not always imply $f = g$. *e.g.,* let,

$$f(x) = 2 \ \forall \ x \in [0, 2] \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{for } x \in [0, 1) \\ 1 & \text{for } x = 1 \\ 2 & \text{for } x \in (1, 2] \end{cases}$$

Then obviously $f \neq g$ but $\int_0^2 | f(u) - g(u) | \, du = 0.$

Example 1.1.10 *Let S be the set of all sequences of real numbers. Let* $x = \{ x_i \}$ *and* $y = \{ y_i \}$ *be any two members of S. Define* $d: S \times S \to R$ *by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{m^i} \frac{| x_i - y_i |}{1 + | x_i - y_i |},$$

m being any integer greater than 1. Show that d is a metric on S.

Solution: Since $\frac{1}{m^i} \cdot \frac{|x_i - y_i|}{1+|x_i - y_i|} < \frac{1}{m^i}$ and $\sum_{i=1}^{\infty}, \frac{1}{m^i}$ is convergent for $m > 1$, $d(x, y)$ is finite. It is clear that $d(x, y) \geq 0$ always and since each term of the right hand infinite series is non-negative, $d(x, y) = 0$ if and only if $|x_i - y_i| = 0 \; \forall \; i = 1, 2, \ldots$ *i.e.,* if and only if $x_i = y_i \; \forall \; i = 1, 2, \ldots$ *i.e.,* if and only if $x = y$.

Moreover, $$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{m^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} = \sum_{i=1}^{\infty} \frac{1}{m^i} \frac{|y_i - x_i|}{1+|y_i - x_i|} = d(y, x).$$

To prove triangle inequality we need use the result that for any two real members u, v

$$\frac{|u+v|}{1+|u+v|} \leq \frac{|u|}{1+|u|} + \frac{|v|}{1+|v|}.$$

Since for $u, v \in$ R, $|u + v| \leq |u| + |v|$, we have $\frac{1}{|u+v|} \geq \frac{1}{|u|+|v|}$

i.e., $$1 + \frac{1}{|u+v|} \geq 1 + \frac{1}{|u|+|v|}$$

i.e., $$\frac{1+|u+v|}{|u+v|} \geq \frac{1+|u|+|v|}{|u|+|v|}$$

i.e., $$\frac{|u+v|}{1+|u+v|} \leq \frac{|u|+|v|}{1+|u|+|v|} = \frac{|u|}{1+|u|+|v|} + \frac{|v|}{1+|u|+|v|} \leq \frac{|u|}{1+|u|} + \frac{|v|}{1+|v|}$$

Now let $x, y, z \in S$ where $x = \{x_i\}$, $y = \{y_i\}$, $z = \{z_i\}$. Choosing $u = x_i - y_i$, $v = y_i - z_i$, we have

$$\frac{|x_i - z_i|}{1+|x_i - z_i|} \leq \frac{|x_i - y_i|}{1+|x_i - y_i|} + \frac{|y_i - z_i|}{1+|y_i - z_i|}$$

Multiplying both sides by $\frac{1}{m^i}$ and then taking summation over i, we get

$$\sum_{i=1}^{\infty} \frac{1}{m^i} \cdot \frac{|x_i - z_i|}{1+|x_i - z_i|} \leq \sum_{i=1}^{\infty} \frac{1}{m^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} + \sum_{i=1}^{\infty} \frac{1}{m^i} \frac{|y_i - z_i|}{1+|y_i - z_i|}$$

i.e., $$d(x, z) \leq d(x, y) + d(y, z).$$

Thus (S, d) is a metric space. ■

Example 1.1.11 Let S_1 denote the set of all bounded sequences of real numbers. If for $x = \{x_i\}$, $y = \{y_i\} \in S_1$, we define $d(x, y) = \sup_i |x_i - y_i|$, then d is a metric on S_1.

Solution: Left to the reader.

Example 1.1.12 Let S_2 denote the set of all convergent sequences of real numbers. For $x = \{x_i\}$, $y = \{y_i\} \in S_2$, define $d(x, y) = \sup_i |x_i - y_i|$. Show that d is a metric on S_2.

Solution: Left to the reader.

Example 1.1.13 *Let l_2 denote the set of all real sequences $\{x_n\}$ for which $\sum_{n=1}^{\infty} |x_n^2| < \infty$ and let*

$d(\{x_n\}, \{y_n\}) = \left\{\sum_{n=1}^{\infty} |x_n - y_n|^2\right\}^{\frac{1}{2}}$. *To show that (l_2, d) is a metric space.*

Solution: Before getting into the act of proving whether d defined above is a metric or not, we should check that $\sum_{n=1}^{\infty} |x_n - y_n|^2 < \infty$. Since $\{x_n\} \in X$ and $\{y_n\} \in X$, $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |y_n|^2 < \infty$. Further by Cauchy-Schwarz inequality, for each $n \in N$,

$$\sum_{k=1}^{n} |x_k y_k| \le \sqrt{\left(\sum_{k=1}^{n} |x_k|^2\right)\left(\sum_{k=1}^{n} |y_k|^2\right)} \le \sqrt{\left(\sum_{k=1}^{\infty} |x_k|^2\right)\left(\sum_{k=1}^{\infty} |y_k|^2\right)} < \infty$$

since the partial sum sequence $\left\{\sum_{k=1}^{n} |x_n\ y_n|\right\}$ is convergent. This implies $\sum_{n=1}^{\infty} |x_n y_n|$ is convergent.

Again $\sum_{n=1}^{\infty} x_n^2$, $\sum_{n=1}^{\infty} y_n^2$ and $\sum_{n=1}^{\infty} x_n y_n$ being all convergent, the series $\sum_{n=1}^{\infty} (x_n - y_n)^2$ is convergent, ensuring that $d(\{x_n\}, \{y_n\})$ is meaningful.

Non-negativity, positive-definiteness and symmetry of $d(.\,,.)$ follow trivially. For triangle inequality it suffices to show that for any $\{x_n\}, \{y_n\}, \{z_n\} \in X$,

$$d(\{x_n\}, \{y_n\}) \le d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$$

From the triangle inequality proved for the Euclidean metric in example 1.1.2, it follows that for every positive integer n,

$$\left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{\frac{1}{2}} \le \left\{\sum_{i=1}^{n} (x_i - z_i)^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{n} (y_i - z_i)^2\right\}^{\frac{1}{2}}$$

Making n approach infinity, it follows that

$$\left\{\sum_{i=1}^{\infty} (x_i - y_i)^2\right\}^{\frac{1}{2}} \le \left\{\sum_{i=1}^{\infty} (x_i - z_i)^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{\infty} (y_i - z_i)^2\right\}^{\frac{1}{2}}$$

$$\therefore \quad d(\{x_n\}, \{y_n\}) \le d(\{x_n\}, \{z_n\}) + d(\{y_n\}, \{z_n\})$$

This completes our proof that $d(.\,,.)$ is a metric. ■

Remark 1.1.5 (*i*) This metric space is known as **Real Numerical Hilbert Space.**

(*ii*) It is a kind of generalisation of the Euclidean metric to the case of sequences, thought of as infinite tuples.

Example 1.1.14 Trivial Metric For a non empty set X let us define $\rho: X \times X \to R$ by

$$\left.\begin{aligned}\rho(x, y) &= 0 \quad \text{if } x = y \\ &= 1 \quad \text{if } x \neq y.\end{aligned}\right\}$$

Solution: Obviously $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$. $\rho(x, y) = \rho(y, x)\ \forall\ x, y \in X$. Also for $x, y, z\ \varepsilon\ X$, if $x = z$ then x may be equal to y or may not.

So $\quad \rho(x, z) = 0 < 2 = 1 + 1 = \rho(x, y) + \rho(y, z) \quad$ if $x \neq y$

and $\quad \rho(x, z) = 0 \leq 0 + 0 = \rho(x, y) + \rho(y, z)$ if $x = y = z$.

Again if $x \neq z$ then $x = y$ and $y = z$ do not hold simultaneously, *i.e.,* either $x \neq y$ or $y \neq z$ or both.

Therefore, $\quad \rho(x, z) = 1 \leq 1 + 0 = \rho(x, y) + \rho(y, z) \quad$ [in case $x \neq y, y = z$]

$\rho(x, z) = 1 \leq 0 + 1 = \rho(x, y) + \rho(y, z) \quad$ [in case $x = y, y \neq z$]

$\rho(x, z) = 1 < 1 + 1 = \rho(x, y) + \rho(y, z) \quad$ [in case $x \neq y, y \neq z$]

Thus ρ satisfies all the conditions to be a distance function. This metric is known as **trivial metric** or **discrete metric.** ■

Example 1.1.15 *Let R^n denote the set of all real n-tuples, i.e. $R^n = \{x \equiv (x_1, x_2, \ldots x_n); x_i \in R\}$. Show that the function $d_p : R^n \times R^n \to R$ defined by*

$$d_p(x, y) = \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{\frac{1}{p}} \quad p \geq 1$$

*is a metric on R^n, known as **Minkowski's metric.** [The set R^n equipped with this metric is called **l_p^n space**].*

Moreover, $\displaystyle \lim_{p \to \infty} d_p(x, y) = \lim_{p \to \infty} \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{\frac{1}{p}} = \max_{1 \leq i \leq n} |x_i - y_i| = d(x, y)$

Solution: The non-negativity, positive-definiteness and symmetry of $d_p(.,.)$ follow easily. The triangle inequality follows from the finite form of Minkowski's inequality, viz.

$$\left(\sum_{i=1}^{n} |\alpha_i + \beta_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |\beta_i|^p\right)^{\frac{1}{p}} ;\ \alpha_i, \beta_i \in R$$

For the second part,

$$d_p(x, y) = \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \max_{1 \leq i \leq n} |x_i - y_i|$$

$$\therefore \quad \lim_{p \to \infty} d_p(x, y) \leq \lim_{p \to \infty} n^{\frac{1}{p}} \max_{1 \leq i \leq n} |x_i - y_i| \text{ as } \lim_{p \to \infty} n^{\frac{1}{p}} = 1$$

Further, $\displaystyle \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \geq \left(\max_{1 \leq i \leq n} |x_i - y_i|^p\right)^{\frac{1}{p}} = \max_{1 \leq i \leq n} |x_i - y_i|$

$\therefore \quad d_p(x, y) \geq \max_{1 \leq i \leq n} |x_i - y_i|\ \forall\ p \geq 1$ and so $\lim_{p \to \infty} d_p(x, y) \geq \max_{1 \leq i \leq n} |x_i - y_i|$

Hence $\displaystyle \lim_{p \to \infty} d_p(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$

Remark 1.1.6 The metric $\underset{1 \le i \le n}{\text{Max}} \mid x_i - y_i \mid$ obtained as the limiting case of Minkowski's metric is indeed the Chebyshev metric appearing in example 1.1.6 Further for $p = 1$ and $p = 2$, Minkowski's metric reduces to rectangular metric of example 1.1.5 and Euclidean metric of example 1.1.2 respectively.

So far we have furnished a variety of examples of metric spaces. We are now in a position to explore some interesting properties of the metric spaces. The results we are going to prove in the next article aim at primarily spanning the family of metric spaces.

1.2 DEEPER PROPERTIES OF METRIC SPACES

(*a*) *In a metric space* (X, d) *if* $x, y, z \in X$, *then* $\mid d(x, z) - d(y, z) \mid \le d(x, y)$.

Proof: Since d is a metric, by the condition ($d3$), we have

$$d(x, z) \le d(x, y) + d(y, z)$$

or $$d(x, z) - d(y, z) \le d(x, y) \qquad ...(1)$$

Again interchanging x and y, we obtain from (1)

$$d(y, z) - d(x, z) \le d(y, x) = d(x, y) \qquad \text{[By } (d2)\text{]}$$

or $$-\{d(x, z) - d(y, z)\} \le d(x, y) \qquad ...(2)$$

Combining (1) and (2) it follows that

$$\mid d(x, z) - d(y, z) \mid \le d(x, y). \blacksquare$$

(*b*)*The convex combination of two metrics* d_1 *and* d_2 *defined on a non-empty set* X *is again a metric.*

Proof: Here we are to show that for any x, y X the function $d(., .)$ defined by :

$$d(x, y) = \lambda d_1(x, y) + (1 - \lambda)d_2(x, y) \text{ ; } 0 \le \lambda \le 1 \text{ is a metric.}$$

Non-negativity, positive-definiteness and symmetry of $d(.,.)$ trivially follow. Since for any x, y, z X,

$$\begin{aligned} d(x, z) + d(z, y) \ \lambda d_1(x, z) &+ (1 - \lambda)d_1 (z, y) + \lambda d_1(z, y) + (1 - \lambda)d_2 (z, y) \\ &= \lambda[d_1(x, y) + d_1 (z, y)] + (1 - \lambda)[d_2(x, z) + d_2(z, y) \\ &\ge \lambda d_1(x, y) + (1 - \lambda)d_2 (x, y) \\ &= d(x, y) \blacksquare \end{aligned}$$

Remark 1.2.1 On the basis of the above one may conclude the set of metrics defined on a non-empty set is a convex set.

(*c*) *If* (X, d) *is a metric space then* $\left(X, \dfrac{d}{1+d}\right)$ *is also a metric space.*

Proof: For $x, y \in X$ let us agree to write $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$

We need to prove that d_1 is a metric on X.

Since $d(x, y) \ge 0$, it is clear that $d_1(x, y) \ge 0$ and $d_1(x, y) = 0$ if and only if $d(x, y) = 0$ *i.e.,* if and only if $x = y$. Thus d_1 satisfies condition ($d1$).

Next if $x, y \in X$, then

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = d_1(y, x).$$

Finally to prove the triangle inequality we consider the function $\varphi(t) = \frac{t}{1+t}$, $t \in R, t > 0$. Obviously $\varphi(t)$ is continuous and derivable for $t > 0$ and

$$\varphi'(t) = \frac{1}{1+t} - \frac{t}{(1+t)^2} = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0, \ \forall \ t > 0.$$

Therefore $\varphi(t)$ is monotonically increasing function for $t > 0$.

Now let $x, y, z \in X$. Since d is a metric, by $(d3)$, $d(x, z) \le d(x, y) + d(y, z)$ where both side of the inequality is positive. Therefore $\varphi(d(x, z)) \le \varphi(d(x, y) + d(y, z))$

i.e.,
$$\frac{d(x,z)}{1+d(x,z)} \le \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$$
$$= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

i.e., $d_1(x, z) \le d_1(x, y) + d_1(y, z)$

This completes the proof.

Note 1. The triangular inequality can also be proved in the following way:

By $(d3)$, $d(x, z) \le d(x, y) + d(y, z)$; $x, y, z \in X$

Now
$$d_1(x, z) - d_1(y, z)$$
$$= \frac{d(x,z)}{1+d(x,z)} - \frac{d(y,z)}{1+d(y,z)}$$
$$= \frac{d(x,z)+d(x,z)\,d(y,z)-d(y,z)-d(x,z)\,d(y,z)}{\{1+d(x,z)\}\{1+d(y,z)\}}$$
$$= \frac{d(x,z)-d(y,z)}{\{1+d(x,z)\}\{1+d(y,z)\}} \le \frac{d(x,y)}{\{1+d(x,z)\}\{1+d(y,z)\}} \quad \text{[By (1)]}$$

Now
$$\{1 + d(x, z)\}\{1 + d(y, z)\} = 1 + d(x, z) + d(y, z) + d(x, z)\, d(y, z)$$
$$\ge 1 + d(x, y) + d(y, z)$$
$$\ge 1 + d(x, y) \quad \text{[By triangle inequality]}$$

$$\therefore \quad d_1(x, z) - d_1(y, z) \le \frac{d(x,y)}{1+d(x,y)} = d_1(x, y)$$

Thus
$$d_1(x, z) \le d_1(x, y) + d_1(y, z).$$

Note 2. $\frac{d}{1+d}$ defined on the non-empty set X is a useful metric in the sense it is a bounded metric irrespective of the choice of metric d. This is one way of having a standard bounded metric defined on X.

(*d*) *If (X, d) be a metric space and* $\phi(x)$, $x \in R$ *is a monotone increasing concave function that vanishes for* $x = 0$, *then* $\phi_0\, d: X \times X \to R$ *is again a metric.*

Proof: As d is a metric, $d(x, y) \ge 0$. Again ϕ being monotone increasing $\phi\,[d(x, y)] \ge \phi(0) = 0 \ \forall \ x, y \in X$. This ensures non-negativity of $\phi_0\, d$

Again $d(x, y) = d(y, x) \Rightarrow \phi\,[d(x, y)] = \phi\,[d(y, x)] \Leftrightarrow (\phi_0\, d)(x, y) = (\phi_0\, d)(y, x)$, implying symmetry property of $\phi_0\, d$.

If $\phi\,[d(x, y)] = \phi\,(0)$, then as ϕ is monotone, $x = y$ follows, proving that $(\phi_0\, d)$ is positive definite.

Finally we prove triangle inequality in the following.

Since f is a concave function, $0 \le \lambda \le 1$

$\phi(\lambda a + (\lambda - 1)b) \ge \lambda\phi(a) + (1 - \lambda)\phi(b)$

$$\phi(p) = \phi\left(\frac{q}{p+q}.0 + \frac{p}{p+q}.(p+q)\right) \ge \frac{q}{p+q}\ \phi(0) + \frac{p}{p+q}\ \phi(p+q) = \frac{p}{p+q}\ \phi(p+q)$$

$$\left(\text{where } a = 0\,;b = (p+q)\,;\lambda = \frac{q}{p+q}\right)$$

$$\phi(p) = \phi\left(\frac{q}{p+q}.0 + \frac{p}{p+q}.(p+q)\right) \ge \frac{q}{p+q}\ \phi(0) + \frac{p}{p+q}\ \phi(p+q) = \frac{p}{p+q}\ \phi(p+q)$$

$$\left(\text{where } a = 0\,;b = (p+q)\,;\lambda = \frac{p}{p+q}\right)$$

$\therefore \quad \phi(p) + \phi(q) \ge \phi(p+q)$

Putting $p = d(x, z)$ and $q = d(z, y)$, we get $\phi(d(x, z)) + \phi(d(z, y)) \ge \phi(d(x, z) + d(z, y)) \ge \phi(d(x, y))$ as d is a metric and ϕ is monotone increasing.

$\therefore \qquad (\phi_0 d)(x, z) + (\phi_0 d)(z, y) \ge (\phi_0 d)(x, y)$

This completes the proof of the fact $\phi_0 d$: X × X → R is metric.

As a special case to this, choose $\phi(t) = \dfrac{t}{1+t}$. (It is monotone increasing concave function that vanishes for $t = 0$)

Hence $\phi(d(x, y)) = \dfrac{d(x,y)}{1+d(x,y)}$ is a metric on X, no matter what $d(.,.)$ is. This envisages that property (d) is a generalisation of property (c).

Theorem 1.2.1 *If (X, d) is a metric space then (X, d_1) is also a metric space where*

$$d_1(x, y) = Min.\ \{1, d(x, y)\},\ x, y \in X.$$

Proof: The proof is similar to the solution of example 1.1.7. ■

Theorem 1.2.2 *Let $X_1, X_2, \ldots X_n$ be metric spaces with underlying metrics $d_1, d_2, \ldots, d_n$ respectively.*

Define $d: \left(\prod\limits_{i=1}^{n} Xi\right) \times \left(\prod\limits_{i=1}^{n} Xi\right) \to R$ as

$$d(x, y) = Max.\ \{d_1(x_1, y_1), d_2(x_2, y_2) \ldots d_n(x_n, y_n)\}$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$; $x_i, y_i \in X_i$, $i = 1, i, \ldots, n$.

Proof: Clearly $d(x, y) \ge 0$ for all $x, y \in \prod\limits_{i=1}^{n} X_i$, since each $d_i(x_i, y_i) \ge 0$, $i = 1, 2, \ldots, n$.

Now if $x, y \in \prod\limits_{i=1}^{n} X_i$ be such that $x = y$, where $x \equiv (x_1, x_2, \ldots, x_n)$, $y \equiv (y_1, y_2, \ldots, y_n)$ then

$$x_i = y_i \qquad \forall\, i = 1, 2, \ldots, n.$$

$\therefore \qquad d_i(x_i, y_i) = 0 \qquad \forall\, i = 1, 2, \ldots, n.$

This implies Max. $\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} = 0$. *i.e.,* $d(x, y) = 0$

Conversely, let $d(x, y) = 0$ for $x, y \in \prod_{i=1}^{n} X_i$.

Then Max. $\{d_1(x_1, y_1), d_2(x_2, y_2), ..., d_n(x_n, y_n)\} = 0$

This implies $d_i(x_i, y_i) = 0 \quad \forall\, i = 1, 2, ..., n,$

$\therefore \qquad x_i = y_i \ \ \forall\, i = 1, 2, ..., n$, yielding $\ x = y$.

If $\ x, y \in \prod_{i=1}^{n} X_i\ $ then $d(x, y) =$ Max. $\{d_1(x_1, y_1), d_2(x_2, y_2), ..., d_n(x_n, y_n)\}$

$= \text{Max. } \{d_1(y_1, x_1), d_2(x_2, y_2), ..., d_n(y_n, x_n)\} = d(y, x).$

Finally, let $x, y, z \in \prod_{i=1}^{n} X_i$ where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ..., y_n)$, $z \equiv (z_1, z_2, ..., z_n)$, $x_i, y_i, z_i \in X_i$, $i = 1, 2, ..., n$.

Now for each i, $d_i(x_i, z_i) \le d_i(x_i, y_i) + d_i(y_i, z_i)$

$\le \text{Max. } \{d_1(x_1, y_1), d_2(x_2, y_2), ..., d_n(x_n, y_n)\}$

$+ \ \text{Max. } \{d_1(y_1, z_1), d_2(y_2, z_2), ..., d_n(y_n, z_n)\}$

The right hand side of the inequality is an upper bound of $d_i(x_i, z_i)$ for each i

$\therefore \quad \text{Max. } \{d_1(x_1, z_1), ..., d_n(x_n, z_n)\} \le \text{Max. } \{d_1(x_1, y_1), ..., d_n(x_n, y_n)\}$

$+ \text{Max. } \{d_1(y_1, z_1), ..., d_n(y_n, z_n)\}$

i.e., $\quad d(x, z) \le d(x, y) + d(y, z)$. ■

Remark 1.2.2 There are alternative ways of defining a product metric on the cartesian product $\prod_{i=1}^{n} X_i$.

One simple alternative is given by $d(x, y) = \sum_{i=1}^{n} d_i(x_i\, y_i)$ where $x = (x_1, x_2, ..., x_n)$; $y = (y_1, y_2, ..., y_n)$ and $x_i, y_i, \ X_i$

Non-negativity, positive-definiteness and symmetry of this function d defined on $\left(\prod_{i=1}^{n} X_i\right) \times \left(\prod_{i=1}^{n} X_i\right)$ follow trivially. Triangle inequality follows from the fact that for each i=1,2,...,n $d_i(.,.)$ satisfies triangle inequality :

$$d(x, y) + d(y, z) = \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(z_i, y_i) = \sum_{i=1}^{n} \left\{ d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(z_i, y_i) \right\} \ge \sum_{i=1}^{n} d_i(x_i, y_i)$$

(c) One can also think of the product of a countably infinite number of metric spaces (X_i, d_i), $i = 1, 2, ...$ but the appropriate product metric would not be the supremum of the factor metrics (c.f. theorem 1.2.2) because supremum of an infinite set is not always finite. It is interesting to see that the standard bounded metric Min $\{1, d_i(x_i, y_i)\}$ corresponding to the factor metrics $d_i(x_i, y_i)$ defined on X_i work as prime ingredients of any product metric. The following theorem illustrates this observation.

Theorem 1.2.3 *Product of acountably infinite number of metric spaces* (X_i, d_i) *i = 1, 2, ... will be a metric space provided we define*

$$d: \left(\prod_{i=1}^{\infty} X_i\right) \times \left(\prod_{i=1}^{\infty} X_i\right) \to R \text{ as } d(x, y) = \prod_{i=1}^{\infty} \frac{1}{2^i} \text{ Min } \{1, d_i,(x_i, y_i)\}$$

where $x = (x_1, x_2, ...)$; $y = (y_1, y_2, ...)$ x_i, y_i, $i = 1, 2, ...$

Proof Min $\{1, d_i (x_i, y_i)\}$ being a standard bounded metric on the component set X_i for any x_i, y_i X_i the series $\sum_{i=1}^{\infty} \frac{1}{2^i} \{1, d_i (x_i, y_i)\}$ dominated by the geometric series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent via comparison test. Hence $d(x, y)$ is well-defined.

Non-negativity, positive definiteness and symmetry of d(.,.) follow easily. For proving triangle inequality we consider three elements $x = (x_1, x_2, ...)$, $y = (y_1, y_2, ...)$ and $z = (z_1, z_2...)$ belonging to $\prod_{i=1}^{\infty} X_i$.

$$\begin{aligned} d(x, y) + d(y, z) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \text{ Min } \{1, d_i(x_i, y_i\} + \sum_{i=1}^{\infty} \frac{1}{2^i} \{1, d_i(y_i, z_i)\} \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} [\text{Min } \{1, d_i(x_i, y_i)\} + \text{Min } \{1, d, (y_i, z_i)\}] \\ &\geq \sum_{i=1}^{\infty} \frac{1}{2^i} \text{ Min } \{1, d_i (x_i, y_i, z_i)\} = d(x, z) \end{aligned}$$

Thus d (.,.) satisfies all properties of a metric. ■

Remark 1.2.3 The geometric series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ dominating the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ Min $\{1, d_i(x_i, y_i)\}$ representing $d(x, y)$ in the above theorem can be substituted by any other convergent positive term series, *e.g.*, the p-series $\sum_{i=1}^{\infty} \frac{1}{i^p}$ $p > 1$. In this case the working function $d : \left(\prod_{i=1}^{\infty} X_i\right) \times \left(\prod_{i=1}^{\infty} X_i\right) \to R$ will be $d(x, y) =$

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \text{ Min. } (1, d_i)\, (x_i, y_i)\}, p > 1$$

In the problem 13 we ask the reader to verify that d (.,.) is a metric.

The standard bounded metric Min $\{1,d_i (x_i, y_i)\}$ associated with the component X_i can be replaced by another standard bounded metric, *viz.* $d_i (x_i, y_i)/(1 + d_i (x_i, y_i))$ so that the working function

$$d : \left(\prod_{i=1}^{\infty} X_i\right) \times \left(\prod_{i=1}^{\infty} X_i\right) \to R \quad \text{becomes} \quad d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(x_i \,.\, y_i)}{1 + d_i(x_i, y_i)}$$

The proof that d(.,.) is a metric is left to the reader (see problem 13)

1.3 CONSTRUCTION OF A METRIC FROM A PSEUDOMETRIC

Definition 1.3.1 A function $\rho: X \times X \to R$ is said to be a **pseudometric** on a non-empty set X iff

(*i*) $\rho(x, y) \geq 0 \ \forall \ x, y \in X$ (non-negativity)

(*ii*) $x = y \Rightarrow \rho(x, y) = 0 \ \forall \ x, y \in X$ but $\rho(x, y) = 0 \Rightarrow x = y \ \forall \ x, y, \in X$

(*iii*) $\rho(x, y) = \rho(y, x) \ \forall \ x, y \in X$ (symmetry)

(*iv*) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \ \forall \ x, y \in X$ (triangle inequality)

The function $\rho: R \times R \to R$ defined by $\rho(x, y) = |x^k - y^k|$, k being a positive integer, satisfies non-negativity, symmetry and triangle inequality. However, if k is odd, ρ defined above is a metric but if k is even, ρ defined above is a pseudometric. Had R been replaced by C, ρ turns out to be a pseudometric for $k \neq 1$ as the highest degree of an irreducible polynomial over complex field is unity.

As a second example of pseudometric space, consider the class $X = C(M)$ of all convergent sequences in an arbitrary metric space (M, d) (see definition 3.1.1). Let $x \equiv \{x_n\}$ and $y \equiv \{y_n\}$ be any two elements of X such that $\lim\limits_{n \to \infty} x_n = 1$ and $\lim\limits_{n \to \infty} y_n = m$.

Define $\rho: X \times X \to R$ by the rule $\rho(x, y) = d(l, m)$. Obviously ρ satisfies non-negativity, symmetry and triangle inequality but is not positive-definite since in (M, d) one can have two different sequences converging to the same limit. (*e, g*., in usual metric space R, $\left\{1 + \frac{1}{n}\right\}$ and both $\left\{1 - \frac{1}{n}\right\}$ converge to limit 1). Hence ρ defined above is a pseudometric and (X, ρ) is a pseudometric space.

On X = $C(M)$ we now define a relation ~ as $x \sim y$ iff $\rho(x, y) = 0 \ \forall \ x, y \in X$.

It is a routine exercise to verify that ~ indeed is an equivalence relation on X and thus partitions it into a number of disjoint equivalence classes—each class characterised by the fact that it contains all sequences converging to the same limit. Let [x] denote the equivalence class containing x $\equiv \{x_n\} \in X$ and $\hat{X}$ denote the family of all the equivalence classes generated by ~. Hence $\hat{X} = \{ [x]: x \in X\}$

Define $\hat{\rho}: \hat{X} \times \hat{X} \to R$ by the rule $\hat{\rho}([x], [y]) = \rho(x, y)$ and observe that is a metric on $\hat{X}$. In this way we construct a metric space $(\hat{X}, \hat{\rho})$ from the pseudometric space (X, ρ).

Following this example as a guideline one can show that every pseudometric space can be always crystallised into a metric space by virtue of defining an equivalence relation on the underlying non-empty set. This artifice is similar to that of extracting an injective map from any arbitrary map defined on some non-empty set.

EXERCISE

1. Let $x, y \in R^n$ where $x \equiv (x_1, x_2, ..., x_n)$, $y \equiv (y_1, y_2, ... y_n)$ $x_i, y_i \in R$, $i = 1, 2, ..., n$. Define $d_1: R^n \times R^n \to R$ by

$$d_1(x, y) = \underset{1 \leq i \leq n}{\text{Max.}} |x_i - y_i|$$

Show that d_1 is a metric on R^n.

Examine whether $d_2(x, y) = \underset{1 \le i \le n}{\text{Min.}} |x_i - y_i|$ is a metric or not.

2. Let S_2 be the class of all convergent sequences of real numbers. For $x = \{x_1, x_2, \ldots\ldots x_n \ldots\ldots\}$ and $y = \{y_1, y_2 \ldots\ldots y_n \ldots\ldots\} \in S_2$, define $d(x, y) = \sup_i |x_i - y_i|$. Then prove that d is a metric on S_2.

3. Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces and let $x \equiv (x_1, x_2)$, $y \equiv (y_1, y_2) \in X_1 \times X_2$.
Define $\rho(x, y) = \text{Min.}\ (\rho_1(x_1, y_1), \rho_2(x_2, y_2))$. Examine whether ρ is a metric on $X_1 \times X_2$.
[**Hints.** ρ is not a metric because Min. $(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) = 0$ does not imply both $\rho_1(x_1, y_1)$ and $\rho_2(x_2, y_2)$ are equal to zero and so (x_1, x_2) may not be equal to (y_1, y_2)].

4. In a metric space (X, d) prove that

$$|d(x_1, y_1) - d(x_2, y_2)| \le d(x_1, x_2) + d(y_1, y_2);\ x_1, x_2, y_1, y_2 \in X.$$

5. In a metric space (X, d) prove that

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n);\ x_1, x_2, \ldots, x_n \in X.$$

6. Let B be the collection of all absolutely convergent series of real numbers. Show that (B, d) is a metric space provided we define $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|$.

7. Let R^2 denote the set of all ordered pairs of real numbers. If for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in R^2 we define

$$d(x, y) = \begin{cases} |x_1| + |y_1| + |x_2 - y_2|, \text{ if } x_2 \ne y_2 \\ |x_1 - y_1| \text{ if } x_2 = y_2 \end{cases}$$

then show that (R^2, d) is a metric space.

8. Show that C^n, the set of all ordered n tuples of complex numbers will form metric spaces w.r.t.' the distance functions defined in examples 1.1.3, 1.1.5 and 1.1.6.

9. (*i*) Show that the set R of real numbers does not form a metric space w.r.t. the function $d: R \times R \to R$ defined by $d(x, y) = |\cos(x - y)|$

(*ii*) For what values of k is the funcion $d(.,.)$ defined by

$$d(x,y) = |x - y|^{\frac{1}{k}},\ k = 1, 2, \ldots..$$

is a metric on the set R of real numbers?

(*iii*) Show that the funcion $d_1(.,.)$ defined by

$$d_1(x, y) = \begin{cases} |x - y| & \text{if } xy > 0 \\ |x - y| - 2 & \text{if } xy < 0 \end{cases}$$

is a metric on the set R \ [-1,1]

10. Consider the set $X = \{f(t): t \in R$ and $\int_a^b |f(t)|^p\, dt < \infty\}$. Define $d: X \times X \to R$ by

$d(x, y) = \left[\int_a^b |f(t) - g(t)|^p\, dt\right]^{\frac{1}{p}}$. Show that (X, d) is a metric space, known as-L_p space.

11. Suppose that (X_i, d_i) is a metric space for $i = 1, 2, ..., m$ and let $X = \prod_{i=1}^{m} X_i$. Show that the function d that associates with every pair of points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ of the set the numbers $d(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i)$ and $d(x, y) = \underset{1 \leq i \leq m}{\text{Max}} \{d_i(x_i, y_i)\}$ are metrics on X.

12. Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that $d: X \times Y \to R$ defined by $d((x_1, y_1), (x_2, y_2))$ $= \{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)\}^{\frac{1}{2}}$ " $(x_1, y_1), (x_2, y_2) \in X \times Y$
is a metric on $X \times Y$ (If in a particular, $X = Y = R$ and d_X , d_Y are usual metrics, then we get the Eucliden metric in R^2.) Show also, by method of induction, that the distance between any two points in Euclidean space R^n can be defined by using above technique.

13. Show that the functions $d(.,.)$ and $\bar{d}$ $(.,.)$ defined on $\left(\sum_{i=1}^{\infty} X_i\right) \times \left(\sum_{i=1}^{\infty} X_i\right)$ by

(i) $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$ $\qquad$ (ii) $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{i^p}$ Min $\{1, d_i \ (x_i, y_i)\}$

proposed in Remark 1.2.3 are metrics.