MATH 357
EXERCISE SHEET 1

- Part [(A)] Decide whether the given functions are metric on the given set.
  (1) $d(x,y) = \sqrt{|x-y|}$ on $\mathbb{R}$.
  (2) $d(x,y) = |x^2 - y^2|$ on $\mathbb{R}$.
  (3) $d(x,y) = |x^3 - y^3|$ on $\mathbb{R}$.
  (4) $d(x,y) = |f(x) - f(y)|$ on $\mathbb{R}$, where f is injective.
  (5) $d(x,y) = |f(x) - f(y)|$ on $\mathbb{R}$, where $f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$.
  (6) $d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ on $\mathbb{R}^2$
  (Note that $x = (x_1, x_2)$ and $y = (y_1, y_2)$).
  (7) $d(x,y) = |x_1 - y_1|^2 + |x_2 - y_2|^2$ on $\mathbb{R}^2$.
  (8) $d(x,y) = |x_1 - y_1|$ on $\mathbb{R}^2$.
  (9) $d(x,y) = \begin{cases} |x_1 - y_1| & \text{if} \quad x_2 = y_2 \\ |x_1| + |x_2 - y_2| + |y_1| & \text{if} \quad x_2 \neq y_2 \end{cases}$ on $\mathbb{R}^2$.
  (10) $d(x,y) = \max\{|x_1 - y_1| + |x_2 - y_2|\}$ on $\mathbb{R}^2$.
  (11) $d(x,y) = \min\{|x_1 - y_1| + |x_2 - y_2|\}$ on $\mathbb{R}^2$.
  (12) $d(x,y) = \displaystyle\sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$ on $\mathbb{R}^2$.
  (13) $d(x,y) = |x^2 - y^2|$ on $[0, \infty)$.
- Part [(B)]Let (X,d) and (Y,ρ) be a metric spaces. Decide whether the given functions form a metric on $X \times Y$.
  (1) $e(x,y) = \sqrt{[d(x_1,y_1)]^2 + [\rho(x_2,y_2)]^2}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
  (2) $p(x,y) = \max d(x_1,y_1), \rho(x_2,y_2)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
  (3) $s(x,y) = d(x_1,y_1) + \rho(x_2,y_2)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
- Part [(C)] Let (X,d) be a metric space. Decide whether the function $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$ $\forall x, y \in X$ forms a metric.
- Part [(D)] Let $C[0,1]$ denote the set of all continuous function from $[0,1]$ to $\mathbb{R}$. Define a function $d(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\}$. Is the pair $(C[0,1], d)$ forms a metric space?
- Part [(E)] Let $C[0,1]$ denote the set of all continuous function from $[0,1]$ to $\mathbb{R}$. Define a function $d(f,g) = \displaystyle\int_0^1 |f(t) - g(t)| dt$. Is the pair $(C[0,1], d)$ forms a metric space?
- Part [(F)] Let $C[0,1]$ denote the set of all continuous function from $[0,1]$ to $\mathbb{R}$. Decide whether the given functions forms a metric on C[0,1].
  (1) $d(f,g) = |f(\frac{1}{2}) - g(\frac{1}{2})|$
  (2) $d(f,g) = \sup\{|f'(x) - g'(x)|\}$